

Schwinger–Dyson equation for quarks in a QCD inspired model

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We discuss formulation of QCD in Minkowski–spacetime and effect of an operator product expansion by means of normal ordering of fields in the QCD Lagrangian. The formulation of QCD in the Minkowski–spacetime allows us to solve a constraint equation and decompose the gauge field propagator in the sum of an instantaneous part, which forms a bound state, and a retarded part, which contains the relativistic corrections. In Quantum Field Theory, if we not start with Lagrangian as normal ordering function of all operator fields, one can make normal ordering by means of the operator product expansion, then the gluon condensate appear. This gives us a natural way of obtaining a dimensional parameter in QCD, which is missing in the QCD Lagrangian. We derive a Schwinger–Dyson equation for a quark, which is studied both numerically and analytically. The critical value of the strong coupling constant $\alpha_s = 4/\pi$, above which a nontrivial solution appears and a spontaneous chiral symmetry breaking occurs, is found. For the sake of simplicity, the considered model describes only one flavor massless quark, but the methods can be used in more general case. The Fourier-sine transform of a function with log-power asymptotic was performed.

I. INTRODUCTION.

Strong interaction physics should be described by Quantum Field Theory (QFT) with the Quantum Chromodynamic (QCD) Lagrangian [1–6]. As it shown [5, 6], the running coupling constant α_s is strong enough at small energies, so that perturbation expansion is not applicable. This is a big problem due to the lack of general methods of non-perturbative calculations.

In order to describe the strong interaction, the phenomenological models were developed that are based directly on experimental data and use partly the QCD knowledge: the QCD sum rules [7–9], the Chiral Perturbation Theory [10–13], the Nambu–Jona-Lasinio model and its generalizations [14–20], bag models [21–23] and others [24, 25]. These models can relatively easily reproduce experimental data. However, they have a number of disadvantages: each of these models works in a certain application area but fail in others, the accuracy of theoretical calculations are limited and often less than the accuracy of modern experimental data. And these models are not true theory of strong interactions. This gives impetus to construct models based directly on QCD, for instance: instanton liquid model [26–29], domain wall network [30–35], various estimations from Schwinger–Dyson equations [36–42].

Thus, there exist various approximations to the theory of strong interactions with their specific simplifications of the QCD. In the QCD researches, one should find answers to the key questions, which are the description of QCD vacuum, spontaneous breaking of chiral symmetry, the absence of color particles (confinement problem), the description of bound states, their masses and decay widths.

We consider the theory of strong interaction at low energy. Our aim is to emphasize the importance of formulation in Minkowski-spacetime and effect of an operator product expansion by means of normal ordering of fields in Lagrangian, and to discuss some consequences of this novel approach.

The formulation of QCD in the Minkowski-spacetime allows us to solve a constraint equation and decompose the gauge field propagator in the sum of an instantaneous part, which forms a bound state, and a retarded part, which contains the relativistic corrections. At the first stage, we should neglect the retarded part and use the instantaneous part to construct the bound state. Then the retarded part gives corrections to the already existing bound state. This idea comes from QED [43] (see also [44–46]), where any attempts of working with the entire propagator do not lead to satisfactory results or the decomposition occurs implicitly. Our approach enable us to cover the both high- and low-energy ranges and find the relation between fundamental QCD parameters and low energy constants.

In QFT, for a Lagrangian with unordered operator fields, one can make normal ordering by means of the operator product expansion. Then the gluon condensate and a low energy effective gluon mass appear. This mechanism gives us a natural and fundamental way of obtaining a dimensional parameter in QCD, which is missing in the QCD Lagrangian. The existence of non-zero condensates directly linked to the conformal anomaly of QCD.

In the next section, we start from QCD Lagrangian and derive an effective action of strong interaction. Then in section III, by using this effective action we obtain the Schwinger–Dyson equation for a quark, which is solved both numerically (in section IV) and analytically (in the subsequent sections). In conclusion, we summarize the obtained results and discuss the prospects of the developed methods. Here, for the sake of simplicity, we intentionally neglect some effects, for example, the considered model describes only one flavor massless quark. While investigating of the Schwinger–Dyson equation, we focus mainly on the question of the spontaneous symmetry breaking. Nevertheless,

all the assumptions made to derive the equation are transparent and well-controlled.

II. EFFECTIVE ACTION FOR THE STRONG INTERACTION.

Let us start with the Quantum Chromodynamics Lagrangian, with number of colors $N_c = 3$ and number of flavors $N_f = 1$:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - A_\mu^a j^{a\mu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \quad (1)$$

where A_μ^a , $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$, ψ , m , and $j^{a\mu} = -g\bar{\psi}\gamma^\mu \frac{\lambda^a}{2}\psi$ are the gluon field, gluon field strength tensor, quark field, quark current mass, and color current of quark, respectively.

An effective action for the meson-like bound state can be derived from the Lagrangian (1). To this end, some *restrictions* and *assumptions* are needed. Below the symbol \bullet is introduced for convenience when we discuss another one assumption or restriction. Some of the restrictions are not principle but imposed in order to not overload the reader by technical calculations. Anyway, in the developed model, we outline main ideas that may be important for correct description of meson-like bound state rather than give a complete description of strong interaction, which certainly remains a tremendous problem.

- First, we choose the frame of reference where the bound state, which we obtain and discuss below, is as whole at rest. Therefore only the *static* problems are considered. We emphasize that the proper choice of the reference frame should be done in Minkowski-spacetime rather than in Euclidian-spacetime. Note that the generalization of this theory to one bound state moving on mass shell [45–49] can easily be done, it is sufficient to rewrite various quantities in the comoving frame of reference.

We fix the gauge

$$\partial_k A_k^a(x) = 0, \quad (2)$$

where $k = 1, 2, 3$ and $a = 1, \dots, 8$ are the space and gluon color indexes, respectively.

The gluon term in the Lagrangian takes the form

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} = \\ = \frac{1}{2}\dot{A}_i^a \dot{A}_i^a - \frac{1}{4}F_{ij}^a F^{a ij} + gf^{abc}(\partial_0 A_i^a)A_0^b A_i^c - \frac{1}{2}A_0^a \partial_i \partial_i A_0^a - gf^{abc}(\partial_i A_0^a)A_0^b A_i^c + \frac{1}{2}g^2 f^{abc} f^{ade} A_0^b A_i^c A_0^d A_i^e. \end{aligned} \quad (3)$$

The third term on the right hand side contains the time derivative and thus can be neglected, because only the static problems are considered, as is noted above.

After quantization, the gluon field A_μ^a becomes an operator field. One can consider the vacuum 2-point correlator

$$\langle 0|A_i^a(x)A_j^b(x)|0\rangle = 2C_g \delta_{ij} \delta^{ab}. \quad (4)$$

- We assume that $C_g \neq 0$ and $C_g < \infty$. Actually C_g depends on the energy, but for simplicity we suppose C_g to be a constant. The constant C_g can be determined from a phenomenology.

The fields in Eq. (4) obeys the condition (2). A question at once arises: “How should the formula (4) be rewritten in other gauges?” The answer is to make a gauge transform to (2), then impose the condition (4), and then make the inverse gauge transform. As is said above, when solving the on-shell bound state problem, we always have a privileged frame of reference, in which this bound state as whole is at rest; therefore, we always have the privileged gauge (2), and thus we define (4) in a gauge-covariant manner in this way. Note that physically privileged reference frame is absent for any scattering problem of quarks and gluons, and one cannot define (4) the same way.

Usually in Quantum Field Theory, a Lagrangian contains only normally ordered operator fields. This is a result of normal ordering of an initial Lagrangian where the above correlator-like terms, arising due to the ordering, are omitted, because they are considered as (infinite) vacuum energy contributions. Keeping these terms, we have after the normal ordering

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} = :\frac{1}{2}\dot{A}_i^a \dot{A}_i^a: - :\frac{1}{4}F_{ij}^a F^{a ij}: + :\frac{1}{2}A_0^a(-\Delta + M_g^2)A_0^a: + \dots \quad (5)$$

The term with M_g comes from the last term of formula (3) and $M_g^2 \equiv 6g^2 C_g N_c$. Here we use the relation $f^{acb} f^{acd} = N_c \delta^{bd}$. The quantity M_g might be interpreted as an effective gluon mass in the gauge (2). This is essentially a model-dependent quantity. In this approach, the gluon mass appears before a perturbation expansion. Phenomenological models in which gluons have nonzero effective mass at small energies have been considered earlier by some authors (see [42, 50–57] and references therein).

- Let us consider the dotted terms in (5) as a perturbation and neglect them. This assumption means that we suggest that bound states are *formed* by only some of the terms which explicitly written in expression (5), while the other terms merely give some *corrections* to the already existing bound states. In the basic model developed in this paper, these terms are neglected. The neglected terms can influence on quantitative characteristics of the bound states, but not their presence, and numerical amount of corrections might be not small due to large value of strong coupling constant.

Substituting (5) without dotted terms into Lagrangian (1), we arrive at the generating functional

$$\mathcal{Z} = \int \mathcal{D}A_\mu^a \delta(\partial_k A_k^a) \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int d^4x \left(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + \frac{1}{2} A_0^a (-\Delta + M_g^2) A_0^a - A_0^a j_0^a + A_i^a j_i^a + \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \right) + i \int d^4x (A_i^a J^{ai} + \bar{\eta} \psi + \bar{\psi} \eta) \right].$$

The source J_0^a is not involved, since the field A_0^a is not dynamical degree of freedom with the gauge (2). This is owing to the fact that the corresponding equation of motion is a constraint [58].

Making integration over A_0^a yields

$$\mathcal{Z} = \int \mathcal{D}A_k^a \delta(\partial_k A_k^a) \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int d^4x \left(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + A_i^a j_i^a + \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \right) - \frac{i}{2} \int d^4x d^4y j_0^a(x) \delta(x^0 - y^0) \frac{1}{4\pi} \frac{e^{-M_g |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} j_0^a(y) + i \int d^4x (A_i^a J^{ai} + \bar{\eta} \psi + \bar{\psi} \eta) \right].$$

The term

$$-\frac{1}{2} \int d^4x d^4y j_0^a(x) \delta(x^0 - y^0) \frac{1}{4\pi} \frac{e^{-M_g |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} j_0^a(y)$$

includes a combination of the Gell-Mann matrices, which may be rewritten in the form

$$\frac{\lambda^{ar_1 r_2}}{2} \frac{\lambda^{as_2 s_1}}{2} = \frac{1}{3} \delta^{r_1 s_1} \delta^{r_2 s_2} + \frac{1}{6} \varepsilon^{tr_1 s_2} \varepsilon^{ts_1 r_2}.$$

- We restrict ourselves to the colorless mesons and so neglect the second term. This term is the diquark channel, which plays a role when baryons are taken into account ("baryon = diquark + quark").

Thus within this approximation, the above term can be rewritten in the form

$$\begin{aligned} & -\frac{1}{2} \int d^4x_1 d^4x_2 d^3\mathbf{y}_1 d^3\mathbf{y}_2 \bar{\psi}_{\alpha_1}^{r_1}(x_1) \psi^{\alpha_2 r_2}(x_2) \delta^{r_1 s_1} \times \\ & \quad \times \underbrace{\gamma^{0\alpha_1}_{\alpha_2} \delta^4(x_1 - x_2) \frac{g^2}{12\pi} \frac{e^{-M_g |\mathbf{x}_1 - \mathbf{y}_2|}}{|\mathbf{x}_1 - \mathbf{y}_2|} \delta^3(\mathbf{y}_1 - \mathbf{y}_2) \gamma^{0\beta_2}_{\beta_1} \delta^{r_2 s_2} \bar{\psi}_{\beta_2}^{s_2}(x_2^0, \mathbf{y}_2) \psi^{\beta_1 s_1}(x_1^0, \mathbf{y}_1)}_{\mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2)} = \\ & = -\frac{1}{2} \int d^4x_1 d^3\mathbf{y}_1 d^4x_2 d^3\mathbf{y}_2 \bar{\psi}_{\alpha_1}^r(x_1) \psi^{\beta_1 r}(x_1^0, \mathbf{y}_1) \mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \psi^{\alpha_2 s}(x_2) \bar{\psi}_{\beta_2}^s(x_2^0, \mathbf{y}_2). \end{aligned}$$

where the above formula is the definition of the operator $\mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2)$, and $\psi^{\beta_1 s_1}(x_1^0, \mathbf{y}_1)$ was shifted to the left. One can see that color indexes r and s have been summed inside pairs $\psi \bar{\psi}$, so the pair $\psi \bar{\psi}$ as whole is colorless.

- Let us treat $\psi^{\alpha s}(x^0, \mathbf{x}) \bar{\psi}^s(x^0, \mathbf{y})$ as a real bilocal field.

The operator $\mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2)$ is symmetrical and has an inverse operator \mathcal{K}^{-1} that can be defined by:

$$\int d^4 x_2 d^3 \mathbf{y}_2 \mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \mathcal{K}_{\beta_2 \alpha_3}^{-1 \alpha_2 \beta_3}(x_2, \mathbf{y}_2; x_3, \mathbf{y}_3) = \delta^4(x_1 - x_3) \delta^3(\mathbf{y}_1 - \mathbf{y}_3) \delta^{\alpha_1}_{\alpha_3} \delta^{\beta_3}_{\beta_1} .$$

This allows us to introduce new bilocal field $\mathcal{M}_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y})$ and make a bosonization transform (Habbard-Stratanovich transform) [59–64]:

$$\begin{aligned} & \exp \left[-\frac{i}{2} \int d^4 x_1 d^3 \mathbf{y}_1 d^4 x_2 d^3 \mathbf{y}_2 \bar{\psi}_{\alpha_1}^r(x_1) \psi^{\beta_1 r}(x_1^0, \mathbf{y}_1) \mathcal{K}_{\beta_1 \alpha_2}^{\alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \psi^{\alpha_2 s}(x_2) \bar{\psi}_{\beta_2}^s(x_2^0, \mathbf{y}_2) \right] = \\ & = \int \mathcal{D}\mathcal{M} \exp \left[\frac{i}{2} \int d^4 x_1 d^3 \mathbf{y}_1 d^4 x_2 d^3 \mathbf{y}_2 \mathcal{M}_{\alpha_1}^{\beta_1}(x_1^0, \mathbf{x}_1, \mathbf{y}_1) \mathcal{K}_{\beta_1 \alpha_2}^{-1 \alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \mathcal{M}_{\beta_2}^{\alpha_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) + \right. \\ & \quad \left. + i \int d^4 x d^3 \mathbf{y} \bar{\psi}_{\alpha}^r(x^0, \mathbf{x}) \psi^{\beta r}(x^0, \mathbf{y}) \mathcal{M}_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) \right] . \end{aligned}$$

Finally the generating functional for effective action of strong interaction takes the form

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}A_k^a \delta(\partial_k A_k^a) \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\mathcal{M} \exp \left[i \int d^4 x \left(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + A_i^a j_i^a + \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \right) + \right. \\ & + \frac{i}{2} \int d^4 x_1 d^3 \mathbf{y}_1 d^4 x_2 d^3 \mathbf{y}_2 \mathcal{M}_{\alpha_1}^{\beta_1}(x_1^0, \mathbf{x}_1, \mathbf{y}_1) \mathcal{K}_{\beta_1 \alpha_2}^{-1 \alpha_1 \beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \mathcal{M}_{\beta_2}^{\alpha_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) + \\ & \left. + i \int d^4 x d^3 \mathbf{y} \bar{\psi}_{\alpha}^r(x^0, \mathbf{x}) \psi^{\beta r}(x^0, \mathbf{y}) \mathcal{M}_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) + i \int d^4 x (A_i^a J^{ai} + \bar{\eta} \psi + \bar{\psi} \eta) \right] . \quad (6) \end{aligned}$$

With the help of this generating functional, one can write down any diagrams for processes of interest.

III. SCHWINGER–DYSON EQUATION.

In what follows we restrict ourselves only to the question of spontaneous symmetry breaking in the theory described by the functional (6). For this purpose, it is convenient to derive and investigate the Schwinger–Dyson (Gap) equation for the quark.

It is difficult to examine the Schwinger–Dyson equation in the general form.

- For the sake of simplicity, we use the Stationary Phase method (that is the Semiclassical approximation). This method simplify the Schwinger–Dyson equation but retain its main properties.

According to this method, we should integrate out the Fermion variables ψ and $\bar{\psi}$ in (6), thus deriving the functional for the action S_{eff}

$$\mathcal{Z} = \int \mathcal{D}A_k^a \delta(\partial_k A_k^a) \mathcal{D}\mathcal{M} e^{i S_{eff}} . \quad (7)$$

Then we arrive at the Schwinger–Dyson equation

$$\frac{\delta S_{eff}}{\delta \mathcal{M}}(A_k^a = 0, \bar{\eta} = 0, \eta = 0, J = 0) = 0 , \quad (8)$$

which gives us the Fermion spectrum inside the bound state [45, 59, 60, 62, 65–68].

We introduce the operator

$$G_{mAM}^{-1 \alpha \beta rs}(x, y) \equiv \left(i \gamma^{\mu \alpha}_{\beta} \delta^{rs} \partial_\mu - m \delta^{\alpha}_{\beta} \delta^{rs} + g A_i^a \gamma^{i \alpha}_{\beta} \frac{\lambda^{ars}}{2} \right) \delta^4(x - y) + \mathcal{M}_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) \delta(x^0 - y^0) \delta^{rs}$$

and define its inverse as

$$\int d^4 y G_{mAM}^{-1 \alpha \beta rs}(x, y) G_{mAM}^{\beta \gamma st}(y, z) = \delta^{\alpha}_{\gamma} \delta^{rt} \delta^4(x - z) .$$

In this notations, formula (7) reads

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}A_k^a \delta(\partial_k A_k^a) \mathcal{D}\mathcal{M} \exp \left[i \int d^4x \left(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aj} \right) + \right. \\ & + \frac{i}{2} \int d^4x_1 d^3\mathbf{y}_1 d^4x_2 d^3\mathbf{y}_2 \mathcal{M}_{\alpha_1}^T \beta_1(x_1^0, \mathbf{x}_1, \mathbf{y}_1) \mathcal{K}^{-1\alpha_1} \beta_2(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \mathcal{M}_{\beta_2}^{\alpha_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) - \\ & \left. - i \int d^4x d^4y \bar{\eta}(x) G_{mAM}(x, y) \eta(y) + \text{tr} \ln G_{mAM}^{-1} + i \int d^4x A_i^a J^{ai} \right]. \end{aligned}$$

Inserting the corresponding S_{eff} into equation (8) we arrive at

$$\int d^4x_2 d^3\mathbf{y}_2 \mathcal{K}^{-1\alpha_1} \beta_2(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) \mathcal{M}_{\beta_2}^{\alpha_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) + i \int dy_1^0 G_{mAM}^{\alpha_1 \beta_1 r}(x_1, y_1) \Big|_{A=0} \delta(x_1^0 - y_1^0) = 0. \quad (9)$$

Below in this article, the solution of this equation is denoted by

$$\mathcal{M}_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) = -\Sigma_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) + m \delta_{\beta}^{\alpha} \delta^3(\mathbf{x} - \mathbf{y}).$$

It is convenient to introduce the operator

$$G_{\Sigma}^{-1\alpha}{}_{\beta}(x, y) \equiv i \gamma^{\mu\alpha}{}_{\beta} \partial_{\mu} \delta^4(x - y) - \Sigma_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) \delta(x^0 - y^0),$$

which, on the stationary solutions obeying Eq. (9), coincides with the earlier introduced operator: $G_{mAM}^{-1\alpha}{}_{\beta}{}^{rs}(x, y) \Big|_{A=0} = G_{\Sigma}^{-1\alpha}{}_{\beta}(x, y) \delta^{rs}$. The inverse operator is defined in the standard manner

$$\int d^4y G_{\Sigma}^{-1\alpha}{}_{\beta}(x, y) G_{\Sigma}^{\beta}{}_{\gamma}(y, z) = \delta^{\alpha}{}_{\gamma} \delta^4(x - z).$$

Acting with the operator \mathcal{K} on the both sides of Eq. (9) and using the above notations, we obtain

$$\Sigma_{\beta_1}^{\alpha_1}(x_1^0, \mathbf{x}_1, \mathbf{y}_1) = m \delta_{\beta_1}^{\alpha_1} \delta^3(\mathbf{x}_1 - \mathbf{y}_1) + 3i \int d^4x_2 d^4y_2 \mathcal{K}_{\beta_1\alpha_2}^{\alpha_1\beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2) G_{\Sigma}^{\alpha_2\beta_2}(x_2, y_2) \delta(x_2^0 - y_2^0). \quad (10)$$

- We are looking for a simplest solution of this equation and adopt the following ansatz

$$\Sigma_{\beta}^{\alpha}(x^0, \mathbf{x}, \mathbf{y}) = \delta_{\beta}^{\alpha} \frac{1}{(2\pi)^{\frac{3}{2}}} M(\mathbf{x} - \mathbf{y}).$$

Due to the isotropy, M is radially symmetric and depends only on $|\mathbf{x} - \mathbf{y}|$.

Making the Fourier transform of equation (10), we have

$$M(\mathbf{p}) \delta^{\alpha_1}{}_{\beta_1} = m \delta_{\beta_1}^{\alpha_1} - i \frac{g^2}{(2\pi)^4} \int d^4q \frac{1}{(\mathbf{p} - \mathbf{q})^2 + M_g^2} \gamma^{0\alpha_1}{}_{\alpha_2} G_{\Sigma}^{\alpha_2\beta_2}(q) \gamma^{0\beta_2}{}_{\beta_1}. \quad (11)$$

In momentum space, the operator G_{Σ}^{-1} can easily be reversed

$$G_{\Sigma}(q) = e^{-\gamma^i \frac{q_i}{|\mathbf{q}|} \varphi(\mathbf{q})} \left(\frac{1}{q_0 + E(\mathbf{q}) - i\varepsilon} \cdot \frac{1 + \gamma^0}{2} + \frac{1}{q_0 - E(\mathbf{q}) + i\varepsilon} \cdot \frac{1 - \gamma^0}{2} \right) e^{\gamma^i \frac{q_i}{|\mathbf{q}|} \varphi(\mathbf{q})} \gamma^0,$$

where we put by definition $E(\mathbf{q}) \equiv \sqrt{M(\mathbf{q})^2 + \mathbf{q}^2}$, and: $\cos 2\varphi(\mathbf{q}) \equiv \frac{M(\mathbf{q})}{E(\mathbf{q})}$. One can see that the time-component q_0 appears in Eq. (11) only through $G_{\Sigma}(q)$ and, hence, can be integrated out.

$$M(\mathbf{p}) = m + \frac{\pi g^2}{(2\pi)^4} \int d^3\mathbf{q} \frac{1}{(\mathbf{p} - \mathbf{q})^2 + M_g^2} \frac{M(\mathbf{q})}{E(\mathbf{q})}.$$

After integrating over the solid angle in 3D momentum space, finally the Schwinger–Dyson equation takes the form

$$M(p) = m + \frac{g^2}{(4\pi)^2} \frac{1}{p} \int_0^{\infty} dq \frac{q M(q)}{\sqrt{M^2(q) + q^2}} \ln \left(\frac{M_g^2 + (p + q)^2}{M_g^2 + (p - q)^2} \right), \quad (12)$$

with $p \equiv |\mathbf{p}|$ and $q \equiv |\mathbf{q}|$ being the absolute values of \mathbf{p} and \mathbf{q} , respectively.

As discussed above, Eq. (8) describes a fermion spectrum inside the bound state. Thus, the physical meaning of $M(p)$ is a running quark mass, and, hence, it should be positive for any momentum. At $p = 0$, the value $M(0)$ corresponds to a constituent quark mass, while the current quark mass is m . One can introduce instead of quark charge g a strong coupling constant $\alpha_s \equiv g^2/(4\pi)$. It is well known in QCD α_s is a running coupling, whose value strongly dependent of the energy scale. Moreover, at low energies, the dependence of momentum $\alpha_s(p)$ can not be calculated from the perturbation theory, which is inapplicable due to the large value of α_s . In the literature, there exist various predictions about the shape of $\alpha_s(p)$ (see, e.g., [25, 37–41, 69] and references therein). Nevertheless in this paper, we assume that α_s is a constant; which is consistent with, as we mention above, neglecting corrections to the bound states; this means in particular neglecting all the loop corrections to α_s , and α_s is really a constant in the framework of this approach. So in a way, the used in this article constant α_s can be understood as an average of the strong coupling $\alpha_s(p)$ over p within a low-momentum range.

- We solve the equation (12) only for $m = 0$, which can be justified by the phenomenology. Indeed, $m \ll M(0)$, because the current mass of light u and d quarks is about 5 MeV. On the other hand, the constituent mass of the same quarks is of order 300 MeV for different models.
- One can demand $M(q) \rightarrow 0$ when $q \rightarrow \infty$. Due to the asymptotic freedom at large momenta, the running quark mass tends to the current mass. Although the existence of the asymptotic freedom in our model is questionable, we do not want to violate it explicitly. In addition, if this restriction is fulfilled then the equation (12) does *not* need any renormalization.

It is convenient to introduce the dimensionless variables $\bar{p} \equiv p/M_g$, $\bar{q} \equiv q/M_g$, and $\bar{M}(\bar{p}) \equiv M(p)/M_g$. In this variables, the Schwinger–Dyson equation (12) takes the form

$$\bar{M}(\bar{p}) = \frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \int_0^\infty d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right). \quad (13)$$

It is obvious that there always exists the solution $\bar{M}(\bar{p}) = 0$. Of course, we are looking for a nontrivial solution of this equation.

An attractive feature of the Schwinger–Dyson equation (13) is that it is controlled only by one external parameter g , which should be fixed from the phenomenology. In particular, it follows from the definition of the dimensionless variables and Eq. (13) that the constituent quark mass is a linear function of M_g and the coefficient of proportionality $c(g)$ depends only on g : $M(0) = c(g) \cdot M_g$.

IV. NUMERICAL SOLUTION OF THE SCHWINGER–DYSON EQUATION.

To solve equation (13) numerically, we use the following algorithm. Let us take a zeroth-order approximation function $\bar{M}_0(\bar{p})$, it is desirable that $\bar{M}_0(\bar{p})$ differs from the solution $\bar{M}(\bar{p})$ not much. Then substituting $\bar{M}_0(\bar{p})$ into the integral in the right-hand side (13) we get $\bar{M}_1(\bar{p})$ in the left-hand side. Then $\bar{M}_1(\bar{p})$ is substituted again, and so on. After a certain number of steps we get, up to the errors of computer calculations, the exact solution $\bar{M}(\bar{p})$ for which the substitution into the right-hand side (13) gives itself. Strictly speaking, we should prove that this algorithm is convergent. We did not try to prove this because in all cases that we calculated this algorithm turned out to be convergent. Moreover, there is no difference in the choice of $\bar{M}_0(\bar{p})$ (see below for details).

In some sense, the convergence of the algorithm can be explained by the stability of the solution under small perturbations. Namely, substituting the function $(1 + \varepsilon)\bar{M}(\bar{p})$, where $\varepsilon \ll 1$, into integral (13) we have up to ε^2 terms

$$\begin{aligned} \frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \int_0^\infty d\bar{q} \frac{\bar{q}(1 + \varepsilon)\bar{M}(\bar{q})}{\sqrt{(1 + \varepsilon)^2 \bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right) &\simeq \\ &\simeq (1 + \varepsilon)\bar{M}(\bar{p}) - \varepsilon \frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \int_0^\infty d\bar{q} \frac{\bar{q} \bar{M}^3(\bar{q})}{(\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2})^3} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right). \end{aligned}$$

The last term is smaller than $\varepsilon \bar{M}(\bar{p})$ and has a minus sign. That is why the obtained expression is closer to the solution $\bar{M}(\bar{p})$ than $(1 + \varepsilon)\bar{M}(\bar{p})$.

After some attempts to solve equation (13) numerically, we have found that there are some things that *should be avoided* at numerical computation:

1. The upper limit of integration must be $+\infty$ and cannot be replaced by finite quantity Λ ; otherwise a strong dependence of the solution form Λ appears.
2. $\bar{M}(+\infty) = 0$, otherwise the integral diverges.
3. It is better to avoid replacing the continuous function $\bar{M}(\bar{p})$ by a discrete table $\bar{M}(\bar{p}_i)$ with fixed numbers of points \bar{p}_i . That is because the value of \bar{M} at the penultimate point (at the last point $\bar{M} = 0$, as it is noted above) depends mainly on the behavior of $\bar{M}(\bar{p})$ between this point and the end point and has a weak dependence on the values of \bar{M} at the other points; the value of \bar{M} at the next to penultimate point depends on the value of \bar{M} at the penultimate point and the behavior of $\bar{M}(\bar{p})$ between these three points, and so on. One cannot approximate the behavior of the function $\bar{M}(\bar{p})$ between two points by a linear segment, otherwise this leads to very low accuracy of numerical calculations. Preferably, $\bar{M}(\bar{p})$ expands in a series of known functions. One may also add that maybe we have more accurate results than in paper [70], where a similar equation was considered numerically and such replacing $\bar{M}(\bar{p})$ by the table $\bar{M}(\bar{p}_i)$ was done.

Put by definition $\bar{M}(-\bar{p}) = \bar{M}(\bar{p})$, then equation (13) can be rewritten in the form

$$\bar{M}(\bar{p}) = \frac{g^2}{2(4\pi)^2} \frac{1}{\bar{p}} \int_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right). \quad (14)$$

Let us define the new function

$$W(\bar{q}) \equiv \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}}. \quad (15)$$

One can easily see that $W(\bar{q})$ has the properties

$$\bar{q} \rightarrow +\infty : \quad W(\bar{q}) \simeq \bar{M}(\bar{q}), \quad (16)$$

$$\bar{q} > 0 : \quad 0 \leq W(\bar{q}) \leq \min(\bar{q}, \bar{M}(\bar{q})), \quad (17)$$

$$W(-\bar{q}) = -W(\bar{q}).$$

New variables can be introduced (where λ – is some parameter)

$$\bar{p} = \lambda \tan\left(\frac{\varphi}{2}\right), \quad \varphi \in (-\pi, \pi),$$

$$\bar{q} = \lambda \tan\left(\frac{\theta}{2}\right), \quad \theta \in (-\pi, \pi).$$

In this variables the Schwinger-Dyson equation takes form

$$\bar{M}(\varphi) = \frac{g^2}{2(4\pi)^2} \int_{-\pi}^{+\pi} \frac{d\theta}{2 \tan \frac{\varphi}{2} \cos^2 \frac{\theta}{2}} \ln \left(\frac{1 + \lambda^2 (\tan \frac{\varphi}{2} + \tan \frac{\theta}{2})^2}{1 + \lambda^2 (\tan \frac{\varphi}{2} - \tan \frac{\theta}{2})^2} \right) W(\theta). \quad (18)$$

On $[-\pi, \pi]$ there is a convenient system of the Fourier series functions:

$$\left\{ \begin{array}{l} \bar{M}(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos(k\varphi) \\ W(\theta) = \sum_{k=1}^{\infty} b_k \cdot \sin(k\theta) \end{array} \right\} \quad \left\{ \begin{array}{l} a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} \bar{M}(\varphi) d\varphi \\ a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} \bar{M}(\varphi) \cos(k\varphi) d\varphi \\ b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} W(\theta) \sin(k\theta) d\theta. \end{array} \right.$$

Using the Fourier series expansion we can avoid all the numerical difficulties which were discussed above. As the Fourier harmonics are periodical functions, it would be better if the area of the fastest change of the function lay

closer to the center of the interval. The point $\theta = \frac{\pi}{2}$ corresponds to $\bar{q} = \lambda$, so it dictates the choice of λ . Of course, before the calculation we do not know what value should be taken; fortunately, the incorrect λ leads only to high inaccuracy and low speed of calculation. Equation (18) now takes the matrix form

$$a_k = A_{kj} b_j, \quad (19)$$

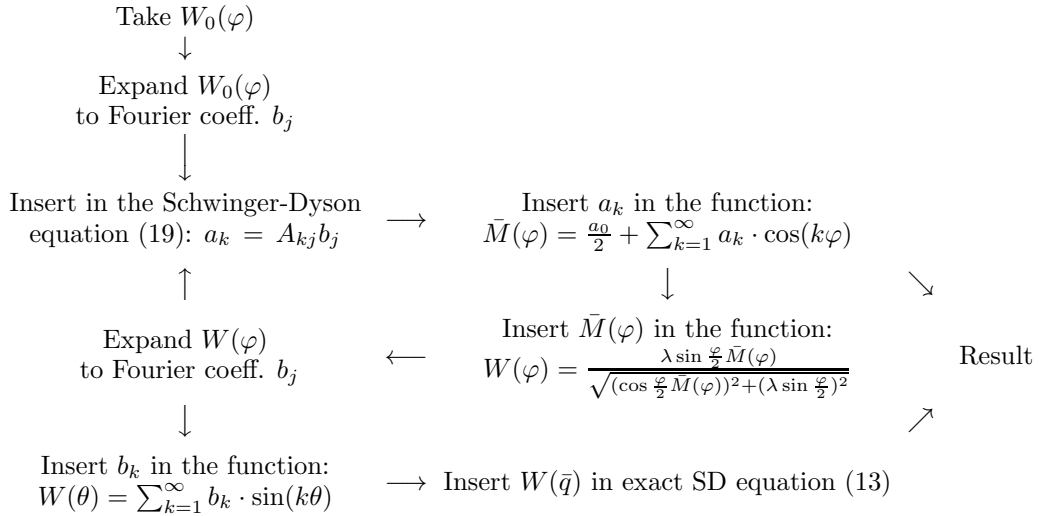
where $A_{kj} \equiv \frac{g^2}{32\pi^3} M_{kj}$, where

$$M_{kj} \equiv \int_{-\pi}^{+\pi} d\varphi \int_{-\pi}^{+\pi} d\theta \frac{\cos(k\varphi)}{2 \tan \frac{\varphi}{2} \cos^2 \frac{\theta}{2}} \ln \left(1 + \frac{\lambda^2 \sin \varphi \sin \theta}{(\cos \frac{\varphi}{2} \cos \frac{\theta}{2})^2 + (\lambda \sin \frac{\varphi-\theta}{2})^2} \right) \sin(j\theta).$$

The matrix M_{kj} contains only the parameter λ and can be calculated separately. Of course in computations, the infinite Fourier series are truncated to finite ones with some number N_h of harmonics.

The zeroth-order approximation function $W_0(\varphi)$ should obey at least condition (17). We tried various $W_0(\varphi)$, which obey (17), and in all cases got the same results differing only in evaluation time. That is why one can take $W_0(\bar{q}) = \bar{q}$.

Finally, the algorithm is the following.



This procedure gives the solution that is expanded into the Fourier series. We can check how well enough it is by substitution in the exact (not matrix (19)) equation (13).

The result of numerical research is the following. There is only the trivial solution $\bar{M}(\bar{p}) = 0$ when $g^2 < 16$. The number 16 is exact and can be obtained from analytical estimations (see Section V). At $g^2 > 16$ a nonzero solution appears. Examples of such calculation are shown in Fig. 1. One can see that the above-mentioned check is successful.

Unfortunately, due to a low accuracy of the numerical calculations, we can not obtain the precise value $\bar{M}(0)$. Namely, equation (13) in the other dimensionless variables $\check{p} \equiv p/M(0)$, $\check{q} \equiv q/M(0)$ and $\check{M}(\check{p}) \equiv M(p)/M(0)$, takes the form

$$\check{M}(\check{p}) = \frac{g^2}{(4\pi)^2} \frac{1}{\check{p}} \int_0^\infty d\check{q} \frac{\check{q} \check{M}(\check{q})}{\sqrt{\check{M}^2(\check{q}) + \check{q}^2}} \ln \left(1 + \frac{4\check{p}\check{q}}{\varkappa^2 + (\check{p} - \check{q})^2} \right),$$

where $\varkappa \equiv M_g/M(0) = 1/\bar{M}(0)$, and there is the condition $\check{M}(0) = 1$. One can see that right-hand side of this equation depends on $M(0)$ only logarithmically, and when $|\check{q} - \check{p}| \gg \varkappa$ it does not depend on $M(0)$ at all.

V. ANALYTICAL RESTRICTIONS.

Using the expression $\bar{M}(\bar{p}) = - \int_{\bar{p}}^{+\infty} d\bar{q} \bar{M}'(\bar{q}) = - \int_0^{+\infty} d\bar{q} \bar{M}'(\bar{q} + \bar{p})$ one can rewrite equation (13) in the form

$$\int_0^{+\infty} d\bar{q} \left(\frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right) + \bar{M}'(\bar{q} + \bar{p}) \right) = 0.$$

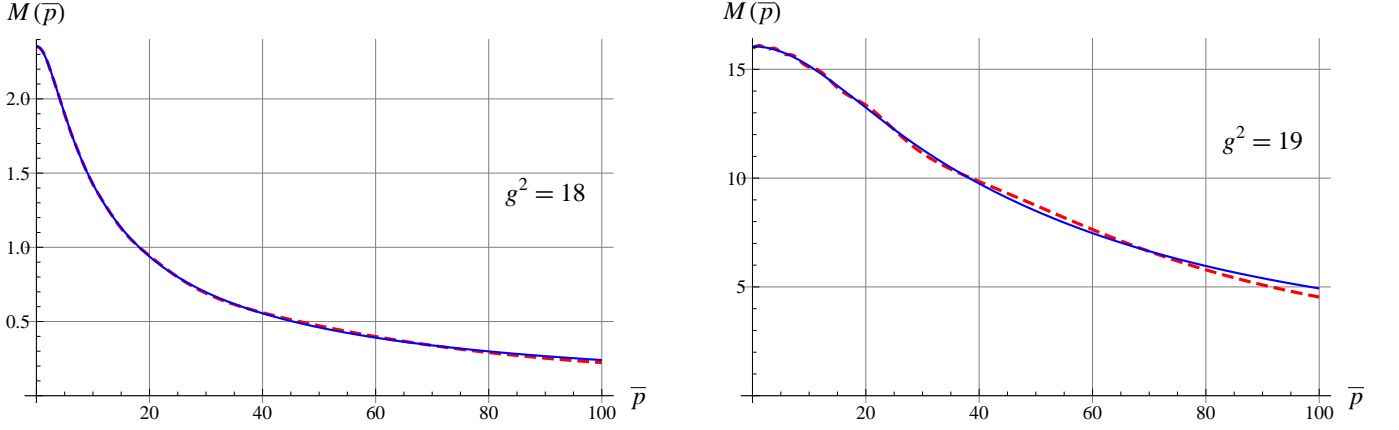


FIG. 1. The running quark mass versus momentum $\bar{M}(\bar{p})$, in units of M_g , at different values of the coupling constant g^2 . The computational parameters are $\lambda = 10$, $N_h = 13$ (see the explanations in the text). The numerical solution of matrix equation (19) is shown in red dashed thick line. The blue solid thin line represents the result of substitution of the previous solution into the right-hand side of Schwinger–Dyson equation (13).

Upon integrating this equation over \bar{p} from zero to infinity, exchange of the order of integration, and direct integration over \bar{p} we have

$$\int_0^{+\infty} d\bar{q} \left(\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{M^2(\bar{q}) + \bar{q}^2}} \frac{g^2}{(4\pi)^2} 2\pi \arctan(\bar{q}) - \bar{M}(\bar{q}) \right) = 0. \quad (20)$$

The formula is correct even if the integral $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q})$ diverges. That is because in all steps the right-hand side of the equation is zero.

If $g^2 \leq 16$, then: $\frac{g^2}{(4\pi)^2} 2\pi \arctan(\bar{q}) < \frac{g^2}{4^2} \leq 1$. So we get

$$0 = \int_0^{+\infty} d\bar{q} \left(\frac{\bar{q}}{\sqrt{M^2(\bar{q}) + \bar{q}^2}} \frac{g^2}{(4\pi)^2} 2\pi \arctan(\bar{q}) - 1 \right) \bar{M}(\bar{q}) < \int_0^{+\infty} d\bar{q} \underbrace{\left(\frac{\bar{q}}{\sqrt{M^2(\bar{q}) + \bar{q}^2}} - 1 \right)}_{<0} \underbrace{\bar{M}(\bar{q})}_{\geq 0}.$$

Which can be satisfied only when $\bar{M}(\bar{p}) = 0$ for any \bar{p} .

For $g^2 > 16$ we have in the limit $\bar{q} \rightarrow \infty$:

$$\left(\frac{\bar{q}}{\sqrt{M^2(\bar{q}) + \bar{q}^2}} \frac{g^2}{(4\pi)^2} 2\pi \arctan(\bar{q}) - 1 \right) \bar{M}(\bar{q}) \rightarrow \left(\frac{g^2}{16} - 1 \right) \bar{M}(\bar{q}).$$

This means that $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q}) < \infty$. For $g^2 > 16$, the integrand in (20) is below zero at small \bar{q} and above zero at large \bar{q} , and so the whole integral (20) can be equal to zero.

Thus, for $g^2 \leq 16$, we have only the trivial solution $M(p) = 0$ of the massless Schwinger-Dyson equation (13). For $g^2 > 16$, the integral $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q})$ is convergent. The threshold value $g^2 = 16$ corresponds to $\alpha_s = \frac{4}{\pi} \simeq 1.27$. Also recall that we have taken $N_f = 1$, for other values of N_f the critical value of α_s might be different. It is worthwhile to notice that this critical value lies near the maximum value $\alpha_s \simeq 1.2$ of the function $\alpha_s(p)$ obtained from the lattice calculations [69].

VI. THE SCHWINGER-DYSON EQUATION IN THE FOURIER SPACE.

In equation (14) in the numerator of the logarithm one can change the variable $\bar{q} \mapsto -\bar{q}$. The Schwinger-Dyson equation (14) then takes the form

$$\bar{M}(\bar{p}) = -\frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \int_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln(1 + (\bar{p} - \bar{q})^2).$$

Let us introduce the notation $\bar{M}_0 \equiv \bar{M}(0)$ and the function:

$$\mathcal{W}(\bar{q}) \equiv \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}_0^2 + \bar{q}^2}}.$$

Then: $\bar{M}(\bar{p}) = \frac{1}{\bar{p}} \sqrt{\bar{M}_0^2 + \bar{p}^2} \mathcal{W}(\bar{p})$. The function $\mathcal{W}(\bar{q})$ has the properties

$$\bar{q} \rightarrow \infty : \quad \mathcal{W}(\bar{q}) \simeq W(\bar{q}) \quad (21)$$

$$\begin{aligned} \bar{q} \rightarrow 0 : \quad \mathcal{W}(\bar{q}) &\simeq W(\bar{q}) \\ \mathcal{W}(-\bar{q}) &= -\mathcal{W}(\bar{q}). \end{aligned} \quad (22)$$

- These properties show that $\mathcal{W}(\bar{p})$ can be determined from a approximate equation:

$$\sqrt{\bar{M}_0^2 + \bar{p}^2} \mathcal{W}(\bar{p}) = -\frac{g^2}{(4\pi)^2} \int_{-\infty}^{+\infty} d\bar{q} \ln(1 + (\bar{p} - \bar{q})^2) \mathcal{W}(\bar{q}). \quad (23)$$

The right-hand side of the last equation can be simplified by means of the Fourier transform

$$\begin{aligned} \mathcal{W}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\bar{p} e^{i\bar{p}x} \mathcal{W}(\bar{p}), & \mathcal{W}(\bar{p}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-i\bar{p}x} \mathcal{W}(x), \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\bar{p} e^{i\bar{p}x} \int_{-\infty}^{+\infty} d\bar{q} \ln(1 + (\bar{p} - \bar{q})^2) \mathcal{W}(\bar{q}) &= \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\bar{q} \int_{-\infty}^{+\infty} d\bar{p} e^{i\bar{q}x} e^{i(\bar{p}-\bar{q})x} \mathcal{W}(\bar{q}) \ln(1 + (\bar{p} - \bar{q})^2) = \\ &= \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\bar{q} e^{i\bar{q}x} \mathcal{W}(\bar{q}) \right) \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\bar{p} e^{i\bar{p}x} \ln(1 + \bar{p}^2) \right) = -2\pi \frac{e^{-|x|}}{|x|} \mathcal{W}(x). \end{aligned}$$

It should be noted that since $\mathcal{W}(\bar{p})$ is an odd function (22), the Fourier transform reduces to a Fourier-sine transform $\mathcal{W}(x) = i\mathcal{W}_s(x)$, where:

$$\mathcal{W}_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} d\bar{p} \sin(\bar{p}x) \mathcal{W}(\bar{p}) \quad \mathcal{W}(\bar{p}) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} dx \sin(\bar{p}x) \mathcal{W}_s(x). \quad (24)$$

The Schwinger-Dyson equation (23) in the Fourier space now takes the form (it is enough to consider $x > 0$):

$$\sqrt{\bar{M}_0^2 - \partial^2} \mathcal{W}_s(x) = \frac{g^2}{8\pi} \frac{e^{-x}}{x} \mathcal{W}_s(x). \quad (25)$$

Using this equation it is convenient to examine the $\bar{p} \rightarrow \infty$ asymptotics. The left-hand side (23) has a simple form $\sqrt{\bar{M}_0^2 + \bar{p}^2} \rightarrow |\bar{p}|$. The right-hand side of the Schwinger-Dyson equation has a simple form in the Fourier space (25). The limit $\bar{p} \rightarrow \infty$ corresponds to the limit $x \rightarrow 0$, so the Taylor expansion can be used.

VII. ASYMPTOTICS OF SOLUTION AT HIGH MOMENTUM.

A. Contribution from low momentum.

Let a be a point such that at $\bar{p} > a$ the solution has the asymptotic behavior. If we consider $\bar{p} \gg a$, the contribution from the right-hand side (13) from a low \bar{q} is

$$\frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \int_0^a d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left(1 + \frac{4\bar{p}\bar{q}}{1 + (\bar{p} - \bar{q})^2} \right) \simeq \frac{g^2}{4\pi^2} \int_0^a d\bar{q} W(\bar{q}) \bar{q} \cdot \frac{1}{\bar{p}^2}.$$

Consequently, (remind that as $\bar{q} \rightarrow +\infty$ from (16) and (21): $\bar{M}(\bar{q}) \simeq W(\bar{q}) \simeq \mathcal{W}(\bar{q})$) the asymptotics of $W(\bar{p})$ cannot be less than $\frac{1}{\bar{p}^2}$:

$$\lim_{\bar{p} \rightarrow +\infty} \frac{1}{\bar{p}^2 W(\bar{p})} < +\infty. \quad (26)$$

Hence, it follows that $W(\bar{p})$ cannot decrease exponentially.

B. Power asymptotics.

One can examine the following ansatz as $\bar{p} \rightarrow \infty$:

$$\mathcal{W}(\bar{p}) = C \text{Sign}(\bar{p}) \frac{1}{|\bar{p}|^\beta} \quad (27)$$

where C is some constant.

β should be real, otherwise the demand $M(p) \geq 0$ is violated. From the requirement of the convergence of the integral $\int d\bar{q} \bar{M}(\bar{q})$ on the upper limit (see Section V) there follows $\beta > 1$. It follows from (26) that $\beta \leq 2$.

The Fourier-sine transform (24) of the function (27) can easily be calculated, for $0 < \beta < 2$:

$$\mathcal{W}_s(x) = C \sqrt{\frac{2}{\pi}} \cos\left(\frac{\beta\pi}{2}\right) \Gamma(1-\beta) \frac{1}{x^{1-\beta}}.$$

Substituting this and (27) in (25) and (23) we get that power asymptotics for $1 < \beta < 2$ is self-consistent if:

$$\frac{1}{g^2} = \frac{1}{8\pi} \frac{\cot\left(\frac{\beta\pi}{2}\right)}{(1-\beta)}. \quad (28)$$

Unfortunately, to obey this formula, one needs $g^2 \leq 16$, which is in contradiction with the results of Section V.

The value $\beta = 2$ may easily be examined and it does not suit too (see Subsection VII D).

Combining all together, we have that power asymptotics (27) is not valid for all β .

C. Log-power asymptotic.

As in the previous subsection we can test a log-power asymptotics as $\bar{p} \rightarrow +\infty$:

$$\mathcal{W}(\bar{p}) \simeq C \frac{(\ln \bar{p})^\gamma}{\bar{p}^\beta}. \quad (29)$$

From the demand $M(p) \geq 0$ it follows that $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. From the requirement of the convergence of the integral $\int d\bar{q} \bar{M}(\bar{q})$ on the upper limit (see Section V) there follows $\beta > 1$, $\gamma \in \mathbb{R}$ or $\beta = 1$, $\gamma < -1$. To obey (26), we need $\beta < 2$, $\gamma \in \mathbb{R}$ or $\beta = 2$, $\gamma \geq 0$.

The Fourier-sine transform (24) of some function with asymptotics (29) can be expressed in terms of elementary functions or relatively simple special functions only in a small number of cases of γ . Fortunately, we do not need the

whole $\mathcal{W}_s(x)$, for our purposes just the asymptotic as $x \rightarrow 0$ is sufficient, and it can be calculated for $0 < \beta < 1$ and $1 < \beta < 2$ and $\gamma \in \mathbb{R}$ (see (A.5) in the Appendix):

$$\mathcal{W}_s(x) \simeq C \sqrt{\frac{2}{\pi}} \cos\left(\frac{\beta\pi}{2}\right) \Gamma(1-\beta) \frac{1}{x^{1-\beta}} \left(\ln \frac{1}{x}\right)^\gamma.$$

This leads to the same constraint (28), so the case $1 < \beta < 2$, $\gamma \in \mathbb{R}$ can not be.

The cases $\beta = 1$, $\gamma < -1$ and $\beta = 2$, $\gamma \geq 0$ can easily be considered directly (see Subsection VII D), and they also do not suit.

Combining the aforesaid we get that log-power asymptotics (29) is not valid for all β and γ .

D. Integral power – Log asymptotics.

Consider asymptotic (29) in the cases $\beta = 1$, $\gamma < -1$ and $\beta = 2$, $\gamma \geq 0$.

Also let a be a point such that for $\bar{p} > a$ the solution $\mathcal{W}(\bar{p})$ has the asymptotic behavior, and we can take $a > 1$. For such \bar{p} the integral in the right-hand side (13) can be decomposed into the sum:

$$\int_0^{+\infty} d\bar{q} W(\bar{q}) \ln\left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2}\right) \simeq \underbrace{\int_0^a d\bar{q} W(\bar{q}) \ln\left(1 + \frac{4\bar{p}\bar{q}}{1 + (\bar{p} - \bar{q})^2}\right)}_{\equiv I_0(\bar{p})} + \frac{C}{\bar{p}^{(\beta-1)}} \int_{\frac{a}{\bar{p}}}^{+\infty} dy \frac{(\ln(\bar{p}y))^\gamma}{y^\beta} \ln\left(\frac{1 + \bar{p}^2(1+y)^2}{1 + \bar{p}^2(1-y)^2}\right)$$

where we introduced a new variable y by the formula $\bar{q} \equiv \bar{p}y$. The asymptotics of $I_0(\bar{p})$ was considered in subsection VII A and it is proportional to $\frac{1}{\bar{p}}$.

Let us choose y_1 and y_2 so that $0 < y_1 \ll 1$ and $1 \ll y_2$. Then the second integral in the right-hand side can be rewritten in the form of the sum: $I_1(\bar{p}) + I_2(\bar{p}) + I_3(\bar{p})$, where

$$\begin{aligned} I_1(\bar{p}) &\equiv \frac{C}{\bar{p}^{(\beta-1)}} \int_{\frac{a}{\bar{p}}}^{y_1} dy \frac{(\ln(\bar{p}y))^\gamma}{y^\beta} \ln\left(1 + \frac{4y}{\frac{1}{\bar{p}^2} + (1-y)^2}\right) \\ I_2(\bar{p}) &\equiv \frac{C}{\bar{p}^{(\beta-1)}} \int_{y_1}^{y_2} dy \frac{(\ln(\bar{p}) + \ln(y))^\gamma}{y^\beta} \ln\left(\frac{1 + \bar{p}^2(1+y)^2}{1 + \bar{p}^2(1-y)^2}\right) \\ I_3(\bar{p}) &\equiv \frac{C}{\bar{p}^{(\beta-1)}} \int_{y_2}^{+\infty} dy \frac{(\ln(\bar{p}y))^\gamma}{y^\beta} \ln\left(1 + \frac{4y}{\frac{1}{\bar{p}^2} + (1-y)^2}\right). \end{aligned}$$

Further, we work only with such \bar{p} that $\frac{a}{\bar{p}} \leq y_1$, $\frac{1}{\bar{p}} \ll y_1$, $y_2 \ll \bar{p}$. For this \bar{p} the integrands are simplified

$$\begin{aligned} I_1(\bar{p}) &\simeq \frac{4C}{\bar{p}^{(\beta-1)}} \int_{\frac{a}{\bar{p}}}^{y_1} dy \frac{(\ln(\bar{p}y))^\gamma}{y^{(\beta-1)}} \\ I_2(\bar{p}) &\simeq \frac{C(\ln(\bar{p}))^\gamma}{\bar{p}^{(\beta-1)}} \int_{y_1}^{y_2} dy \frac{1}{y^\beta} \ln\left(\frac{1 + \bar{p}^2(1+y)^2}{1 + \bar{p}^2(1-y)^2}\right) \\ I_3(\bar{p}) &\simeq \frac{4C}{\bar{p}^{(\beta-1)}} \int_{y_2}^{+\infty} dy \frac{(\ln(\bar{p}y))^\gamma}{y^{(\beta+1)}}. \end{aligned}$$

This integrals can now be calculated directly.

For $\beta = 1$, $\gamma < -1$ this leads to:

$$I_1(\bar{p}) \simeq 4C y_1 (\ln \bar{p})^\gamma \quad I_2(\bar{p}) \simeq C \left(\pi^2 - 4y_1 - \frac{4}{y_2} \right) (\ln \bar{p})^\gamma \quad I_3(\bar{p}) \simeq \frac{4C}{y_2} (\ln \bar{p})^\gamma.$$

Combining all together the right-hand side of (13) equals: $C \frac{g^2}{16} \frac{(\ln \bar{p})^\gamma}{\bar{p}}$, which can be consistent with the left-hand side of (13) only if $g^2 = 16$, but this value is forbidden by the arguments of Section V. Thus, the case $\beta = 1$, $\gamma < -1$ does not suit.

For $\beta = 2$, $\gamma \geq 0$ the integration leads to

$$I_1(\bar{p}) \simeq \frac{4C}{1+\gamma} \frac{(\ln \bar{p})^{\gamma+1}}{\bar{p}} + 4C \ln(y_1) \frac{(\ln \bar{p})^\gamma}{\bar{p}} \quad I_2(\bar{p}) \simeq C \left(4 - 4 \ln(y_1) - \frac{2}{y_2^2} \right) \frac{(\ln \bar{p})^\gamma}{\bar{p}} \quad I_3(\bar{p}) \simeq C \frac{2}{y_2^2} \frac{(\ln \bar{p})^\gamma}{\bar{p}}.$$

From the aforesaid

$$I_0(\bar{p}) + I_1(\bar{p}) + I_2(\bar{p}) + I_3(\bar{p}) \simeq \frac{4C}{1+\gamma} \frac{(\ln \bar{p})^{\gamma+1}}{\bar{p}} + 4C \frac{(\ln \bar{p})^\gamma}{\bar{p}} + \frac{4}{\bar{p}} \int_0^a d\bar{q} W(\bar{q}) \bar{q}. \quad (30)$$

We can see that the right- and left-hand sides of (13) here are not self-consistent. So the case $\beta = 2$, $\gamma \geq 0$ is not valid either.

The $I_0(\bar{p})$ always gives the contribution to asymptotics proportional to $\frac{1}{\bar{p}^2}$. After substitution this asymptotics into the right-side of (13), according to (30), this should lead to a contribution proportional to $\frac{\ln \bar{p}}{\bar{p}^2}$; after substitution the last one we should get $\frac{(\ln \bar{p})^2}{\bar{p}}$, and so on. Consequently, we can conclude that condition (26) can be generalized to:

$$\lim_{\bar{p} \rightarrow +\infty} \frac{(\ln \bar{p})^\gamma}{\bar{p}^2 W(\bar{p})} < +\infty$$

where $\gamma \in \mathbb{R}$.

Furthermore, the form of (30) suggests that the solution should be searched in the form of a series in powers of the logarithm.

E. Series asymptotics.

We can suppose that asymptotics of the solution of equation (13) as $\bar{p} \rightarrow +\infty$ has the form

$$W(\bar{p}) = C_0 \frac{1}{\bar{p}^\beta} + C_1 \frac{\ln(\bar{p})}{\bar{p}^\beta} + C_2 \frac{(\ln(\bar{p}))^2}{\bar{p}^\beta} + \dots + L(\bar{p}), \quad (31)$$

where $1 < \beta < 2$, the function $L(\bar{p})$ decreases faster than $\frac{1}{\bar{p}^2 \ln(\bar{p})}$, and the series does not reduce to powers of \bar{p} or $\ln(\bar{p})$.

In the left-hand side of (13), if we neglect $L(\bar{p})$, then with the same accuracy $\bar{M}(\bar{q}) \simeq W(\bar{q})$, this comes from the formula inverse to (15).

In the right-hand side (13), the integral can be expanded into the sum: $\int_0^\infty = \int_0^a + \int_a^\infty$, the integral \int_0^a was considered in subsection VII A, the other integral (to the accuracy of $L(\bar{p})$, which can give not more than $\frac{1}{\bar{p}}$ asymptotic) is

$$\int_a^\infty d\bar{q} W(\bar{q}) \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right) \simeq \left(C_0 + C_1 \left(-\frac{\partial}{\partial \beta} \right) + C_2 \left(-\frac{\partial}{\partial \beta} \right)^2 + \dots \right) \int_a^\infty d\bar{q} \frac{1}{\bar{q}^\beta} \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right).$$

The right-hand side integral can also be expanded into the sum: $\int_a^\infty = \int_0^\infty - \int_0^a$. The \int_0^a gives the asymptotics $\frac{1}{\bar{p}}$ for the same reasons as in the low momentum case (see subsection VII A). The integral \int_0^∞ can easily be evaluated by parts

$$\begin{aligned} \int_0^\infty d\bar{q} \frac{1}{-\beta+1} \left(\frac{\partial}{\partial \bar{q}} \bar{q}^{-\beta+1} \right) \ln \left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right) &= \frac{2}{\beta-1} \int_0^\infty d\bar{q} \frac{1}{\bar{q}^{\beta-1}} \left(\frac{\bar{p} + \bar{q}}{1 + (\bar{p} + \bar{q})^2} + \frac{\bar{p} - \bar{q}}{1 + (\bar{p} - \bar{q})^2} \right) = \\ &= \frac{2\pi}{\beta-1} \frac{(1 + \bar{p}^2)^{-\frac{\beta}{2}}}{\sin(\beta\pi)} \left((\bar{p}^2 - 1) \sin(\beta \operatorname{arccot}(\bar{p})) - 2\bar{p} \cos(\beta \operatorname{arccot}(\bar{p})) - \bar{p}^2 \sin(\beta\pi - (-2+\beta) \operatorname{arccot}(\bar{p})) \right) + \\ &\quad + 2\bar{p} \cos\left(\frac{\beta\pi}{2}\right) \left(\cos(\beta \arctan(\bar{p})) + \bar{p} \sin(\beta \arctan(\bar{p})) \right) + \sin\left(\frac{\beta\pi}{2} + (-2+\beta) \arctan(\bar{p})\right). \end{aligned}$$

As $\bar{p} \rightarrow +\infty$ this leads to:

$$\int_0^\infty d\bar{q} \frac{1}{\bar{q}^\beta} \ln\left(\frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2}\right) = 2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1} \frac{1}{\bar{p}^{\beta-1}} + h(\bar{p}, \beta) \frac{1}{\bar{p}^\beta}$$

where $\lim_{\bar{p} \rightarrow +\infty} h(\bar{p}, \beta) < +\infty$, so this term can be neglected.

Thus, from various sources the term $A \frac{1}{\bar{p}^2}$ appears in the right-hand side of the equation. We rewrite this term in the form:

$$A \frac{1}{\bar{p}^2} = A \frac{1}{\bar{p}^\beta} + A(\beta-2) \frac{\ln(\bar{p})}{\bar{p}^\beta} + A \frac{(\beta-2)^2}{2!} \frac{(\ln(\bar{p}))^2}{\bar{p}^\beta} + \dots$$

The absence of this term is the reason why asymptotics (27) and (29) are not valid. This term appears from a middle values of \bar{q} in the integral in the right-hand side of equation (13).

Finally, the substitution of asymptotics (31) in equation (13) leads to the infinite matrix equation

$$\begin{aligned} \frac{(4\pi)^2}{g^2} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \end{pmatrix} &= \begin{pmatrix} A \\ A(\beta-2) \\ A \frac{(\beta-2)^2}{2!} \\ \vdots \end{pmatrix} + \\ &+ \begin{pmatrix} 2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1} & \left(-\frac{\partial}{\partial\beta}\right) \left(2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1}\right) & \left(-\frac{\partial}{\partial\beta}\right)^2 \left(2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1}\right) & \dots \\ 0 & \binom{1}{1} 2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1} & \binom{2}{1} \left(-\frac{\partial}{\partial\beta}\right) \left(2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1}\right) & \dots \\ 0 & 0 & \binom{2}{2} 2\pi \frac{\tan\left(\frac{(\beta-1)\pi}{2}\right)}{\beta-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \end{pmatrix} \end{aligned}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Likely, one cannot reduce the infinite matrix and columns to finite ones, because the elements in the rows in the matrix do not decrease. This is one of the reasons why asymptotics (31) is permitted while asymptotics (27) and (29) are not. The other reason is that in the evaluation of (31) we do not neglect the term $A \frac{1}{\bar{p}^2}$.

VIII. CONCLUSIONS.

The natural method of obtaining a dimensional parameter in QCD is suggested by means of normal ordering of fields in the Lagrangian. Within our model, the dimensional parameter QCD is nothing else but the effective gluon mass.

Based on QCD the effective action of strong interaction (6) was constructed.

In the framework of the constructed model, the Schwinger–Dyson equation with the effective gluon mass (13) is investigated both analytically and numerically. It is shown that spontaneous chiral symmetry breaking occurs and a nontrivial dependance of the quark mass on momenta. The critical value of the strong coupling constant (equals to $\alpha_s = 4/\pi$), *above* which the spontaneous breaking occurs, is found in the semiclassical approximation. It is proved strictly that *below* this critical value, the Schwinger–Dyson equation has only trivial non-negative solution $M(p) = 0$. In the paper, we do not study analytically uniqueness of the nontrivial solution above the critical value α_s .

Although the derivation of the effective action of strong interaction (6) from the QCD Lagrangian (1) is clear and well-controlled, numerous assumptions are done during the derivation. Thus, the obtained results are qualitative rather

than quantitative. Taking into account the neglected terms could amend the model and make it quite quantitative. For better understanding of the solution of the Schwinger–Dyson equation in the region of the large coupling constant, it would be better to improve the numerical computation scheme. For instance, the accuracy of the numerical simulations is likely to be insufficient to obtain the precise value of $M(p)$ at $p = 0$.

The developed analytical methods of solving and analyzing as well as created programs for numerical calculation the Schwinger–Dyson equation can be used not only in the considered specific kernel but for various other kernels.

The Fourier-sine transform of function (A.1) with log-power asymptotic was performed (A.4), and the leading asymptotic was found.

In the papers [43, 47] (see also [48, 62, 67, 68] and references therein) it is shown how the Bethe–Salpeter equation, which describe the spectrum and wave functions of the bound states, can be derived in the framework of the Stationary Phase method. To cope with this Bethe–Salpeter equation, one should already have the solution of the corresponding Schwinger–Dyson equation (8) as the “input function”. The investigation of the Bethe–Salpeter equation is beyond the scope of this paper and will be done later.

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Appendix: Fourier-sine transform of a log-power function.

Consider the function

$$F_a(x, \gamma, \beta) \equiv \int_0^{+\infty} d\bar{p} \frac{(\ln(a+\bar{p}))^\gamma}{(a+\bar{p})^\beta} \sin(x\bar{p}) , \quad (\text{A.1})$$

where: $a > 1$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$. As: $F_a(-x, \gamma, \beta) = -F_a(x, \gamma, \beta)$, below we will take: $x > 0$. Our aim is to find the asymptotic behavior of $F_a(x, \gamma, \beta)$ at $x \rightarrow 0$.

This function has the property

$$F_a(x, \gamma, \beta) = -\frac{d}{d\beta} F_a(x, \gamma-1, \beta) , \quad (\text{A.2})$$

so the practically interesting case is: $-1 < \gamma \leq 0$. Let us introduce the new notation: $\bar{\gamma} \equiv -\gamma$, where: $0 \leq \bar{\gamma} < 1$.

Using the formula

$$\frac{1}{(\ln(a+\bar{p}))^{\bar{\gamma}}} = \int_0^{+\infty} dt \frac{\sin(t \ln(a+\bar{p}))}{t^{1-\bar{\gamma}}} \frac{1}{\sin(\frac{\pi\bar{\gamma}}{2}) \Gamma(\bar{\gamma})}$$

and exchanging the order of integrations we get

$$F_a(x, -\bar{\gamma}, \beta) = \frac{1}{\sin(\frac{\pi\bar{\gamma}}{2}) \Gamma(\bar{\gamma})} \int_0^{+\infty} dt \frac{1}{t^{1-\bar{\gamma}}} \int_0^{+\infty} d\bar{p} \frac{\sin(x\bar{p})}{(a+\bar{p})^\beta} \sin(t \ln(a+\bar{p})) . \quad (\text{A.3})$$

One can expand the function: $\sin(t \ln(a+\bar{p})) = \sin(t \ln(x(a+\bar{p})) + t \ln \frac{1}{x})$ into the series around the point $t \ln \frac{1}{x}$. Put by definition:

$$H_a(x, \beta) \equiv \int_0^{+\infty} d\bar{p} \frac{\sin(x\bar{p})}{(a+\bar{p})^\beta} ,$$

we arrive at

$$\begin{aligned} & \int_0^{+\infty} d\bar{p} \frac{\sin(x\bar{p})}{(a+\bar{p})^\beta} \sin\left(t \ln(a+\bar{p})\right) = \\ & = x^\beta \sin\left(t \ln \frac{1}{x}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{d^{2k}}{d\beta^{2k}} \frac{H_a(x, \beta)}{x^\beta} \right) t^{2k} - x^\beta \cos\left(t \ln \frac{1}{x}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{d^{2k+1}}{d\beta^{2k+1}} \frac{H_a(x, \beta)}{x^\beta} \right) t^{2k+1}. \end{aligned}$$

Substituting the latter into (A.3) and changing the variable: $\bar{t} \equiv t \ln \frac{1}{x}$, we finally have

$$\begin{aligned} F_a(x, -\bar{\gamma}, \beta) = & \frac{1}{\sin\left(\frac{\pi\bar{\gamma}}{2}\right) \Gamma(\bar{\gamma})} \frac{x^\beta}{\left(\ln \frac{1}{x}\right)^{\bar{\gamma}}} \int_0^{+\infty} d\bar{t} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{d^{2k}}{d\beta^{2k}} \frac{H_a(x, \beta)}{x^\beta} \right) \frac{1}{\left(\ln \frac{1}{x}\right)^{2k}} \bar{t}^{2k-1+\bar{\gamma}} \sin \bar{t} - \right. \\ & \left. - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{d^{2k+1}}{d\beta^{2k+1}} \frac{H_a(x, \beta)}{x^\beta} \right) \frac{1}{\left(\ln \frac{1}{x}\right)^{2k+1}} \bar{t}^{2k+\bar{\gamma}} \cos \bar{t} \right). \quad (\text{A.4}) \end{aligned}$$

Further steps strongly depend on which β we want to consider and how many terms of the series we are interested in. In our case, we are interested in only leading asymptotics and values: $0 < \beta < 1$ or $1 < \beta < 2$. For these β :

$$H_a(x, \beta) \simeq \frac{\pi \cos\left(\frac{\beta\pi}{2}\right)}{\Gamma(\beta) \sin(\beta\pi)} \frac{1}{x^{1-\beta}}$$

and leading asymptotics comes from the $k = 0$ term in the first sums (A.4). Thus, we get

$$F_a(x, -\bar{\gamma}, \beta) \simeq \frac{\pi \cos\left(\frac{\beta\pi}{2}\right)}{\Gamma(\beta) \sin(\beta\pi)} \frac{x^{\beta-1}}{\left(\ln \frac{1}{x}\right)^{\bar{\gamma}}}.$$

Recollecting (A.2) we finally have

$$F_a(x, \gamma, \beta) \simeq \Gamma(1-\beta) \cos\left(\frac{\beta\pi}{2}\right) x^{\beta-1} \left(\ln \frac{1}{x}\right)^\gamma, \quad (\text{A.5})$$

where: $x > 0$, $0 < \beta < 1$ or $1 < \beta < 2$, $\gamma \in \mathbb{R}$. The coefficient does not depend on γ , $a > 1$ and a is absent in the right-hand side, as it should be.

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